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A constructive approach for an optimal control applied to a class of nonlinear time delay systems

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ABSTRACT

In this contribution, we obtain a nonlinear controller for a class of nonlinear time delay systems, by using the inverse optimality approach. We avoid the solution of the Hamilton Jacobi Bellman type equation and the determination of the Bellman's functional by extending the inverse optimality approach for delay free nonlinear systems to time delay nonlinear systems. This is achieved by combining the Control Lyapunov Function framework and Lyapunov-Krasovskii functionals of complete type. Explicit formulas for an optimal control are obtained. The efficiency of the proposed method is illustrated via experimental results applied to a dehydration process whose model includes a delayed state linear part and a delayed nonlinear part. To give evidence of the good performance of the proposed control law, experimental comparison against an industrial Proportional Integral Derivative controller and optimal linear controller. Additionally experimental robustness tests are presented.

1. Introduction

In the inverse optimality approach [1] for delay free nonlinear systems, the existence of a Control Lyapunov Function (CLF) [2] is instrumental in obtaining the optimal control. If the CLF is known, an assumption which is not always fulfilled, it can be used as a Bellman function to obtain explicit formulas of the nonlinear controller, thus avoiding the solution of the Hamilton Jacobi Bellman equation (HJB).

For time delay systems, the inverse optimality problem has been studied in a number of contributions. In [3], a feedback compensation for linear time invariant systems with delayed input is presented by using a Lyapunov-Krasovskii functional of prescribed form. In [4], the assumption of the existence of a Control Lyapunov-Krasovskii Functional (CLKF) is made and a solution to the inverse optimality problem is found, leading in some cases to the construction of stabilizing control laws of explicit form. Lyapunov-Krasovskii functionals and Lyapunov-Razumikhin functions are also used in the stabilization problem of nonlinear time delay systems, see [5–7].

More recently, complete type functionals [8], whose form is known whenever the nominal linear time delay system is stable, have been successfully used for synthesizing controllers. In [9], a suboptimal controller for stable linear systems with pointwise and distributed delays is designed. A guaranteed cost control strategy is proposed for the cases of linear time delay systems with structured disturbances in [10], and of a class of nonlinear systems in [11].

In view of the above ideas, it seems natural to use complete type functionals in the inverse optimality approach. We consider here nonlinear time delay systems that consist of a linear part which is exponentially stable and nonlinear disturbances satisfying a Lipschitz condition. The first step is to find the sufficient condition under which the complete type functional is a CLKF and the second is to extend the delay free inverse optimality approach. As a consequence, we find the optimal controller that minimizes a performance index in the sense of inverse optimality, without solving a Bellman type equation.

The obtained controller is experimentally tested on an open loop stable dehydration prototype. The nonlinearities in the model are represented by a polynomial equation that depends on the process variables [12]. It is compared with an industrial Proportional Integral Derivative (PID) controller, in terms of power consumption. Notice that in the industry of developed countries, drying is an energy intensive

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process that represents from 10% to 25% of energy consumption [13], hence for optimal drying conditions, an optimal controller is preferred to decrease drying time and fuel consumption [14].

The contribution is organized as follows. In Section 2, we briefly remind the fundamental ideas of the Lyapunov-Krasovskii complete type functional approach, the definition of CLKF and inverse optimality problem for delay free nonlinear systems. Next, we formulate the problem and the class of considered nonlinear time delay systems. The main results are presented in Section 3, where a sufficient condition which guarantees that the complete type functional is a CLKF for a class of delayed nonlinear systems is found, and the explicit nonlinear optimal control law is synthesized. Experimental results on a dehydration process illustrate the efficiency of our controller in Section 4. Finally, the conclusions are discussed in Section 5.

Notation: *The function* $x_t = x(t+\theta)$, $\theta \in [-h, 0]$, *is the restriction of the solution to the interval* $[t-h, t]$. *C is a space* $\mathcal{C}([-h, 0], \mathbb{R}^n)$ of $\mathbb{R}^n$*-valued continuous functions on* $[-h, 0]$. $\mathcal{PC}([-h, 0], \mathbb{R}^n)$ *is the space of piecewise* $\mathbb{R}^n$*-valued continuous functions on* $[-h, 0]$ *and it is supplied with the standard uniform norm [15]*:

$$\|\varphi\|_h = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|.$$

The Euclidean norm $\|\cdot\|$ *is used for vectors and the corresponding induced norm for matrices. A continuous scalar function* α *is said to belong to class* $\mathcal{K}$ *if it is strictly increasing and* $\alpha(0)=0$; α *is said to belong to class* $\mathcal{K}_\infty$ *if, in addition,* $\alpha(s) \to \infty$ *as* $s \to \infty$ *[16].* $\mathcal{U}$ *denotes a set of admissible values of the control variable u which are piecewise continuous functions.*

2. Preliminary results

In this section we review the main concepts and definitions of complete type Lyapunov-Krasovskii functionals, Control Lyapunov Functions, and the scheme of inverse optimality for delay free nonlinear systems.

2.1. Lyapunov-Krasovskii functionals of complete type

We recall some key concepts and results on complete type functionals [8], where the Lyapunov matrix for time delay systems $U(\tau)$ (see [17]) plays an essential role.

Consider time delay systems of the form:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad t \geq 0, \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$, $h > 0$ is the known delay, and $A_0, A_1 \in \mathbb{R}^{n\times n}$ are real matrices. Let $\varphi : [-h, 0] \to \mathbb{R}^n$ be an initial function. We assume that the function φ belongs to the space $\mathcal{PC}([-h, 0], \mathbb{R}^n)$, of piecewise continuous functions defined on the segment $[-h, 0]$.

$$x(\theta) = \varphi(\theta), \quad \theta \in [-h, 0].$$

The explicit form of the complete type functional $V : \mathcal{PC}([-h, 0], \mathbb{R}^n) \to \mathbb{R}$ is presented in the following theorem.

Theorem 1 ([8]). *Given three symmetric matrices* W_j, $j = 0, 1, 2$, *let us define the functional*

$$w(\varphi) = \varphi^T(0)W_0\varphi(0) + \varphi^T(-h)W_1\varphi(-h) + \int_{-h}^{0} \varphi^T(\theta)W_2\varphi(\theta)d\theta.$$

If there exists a Lyapunov matrix $U(\tau)$ *associated with the matrix* $W = W_0 + W_1 + hW_2$, *the functional*

$$\begin{aligned} V(\varphi) &= \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)d\theta + \int_{-h}^{0} \varphi^T(\theta)\left[W_1 + \left(h+\theta\right)W_2\right]\varphi(\theta)d\theta \\ &\quad + \int_{-h}^{0} \varphi^T(\theta_1)A_1^T\left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2\right]d\theta_1, \end{aligned} \tag{2}$$

has time derivative along the solutions of system (1) given by

$$\frac{dV(x_t)}{dt} = -w(x_t), \quad t \geq 0.$$

Here $U(\tau)$, *the Lyapunov matrix of system (1) associated to* $W = W_0 + W_1 + hW_2$ *is the unique solution of the following equalities [8,17]:*

$$\begin{aligned} &U(-\tau) = U^T(\tau), \quad \tau \geq 0, && (\textit{Symmetry property}) \\ &\frac{d}{dt}U(\tau) = U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \geq 0, && (\textit{Dynamic property}) \\ &-W = U(0)A_0 + U(-h)A_1 + A_0^TU(0) + A_1^TU(h). && (\textit{Algebraic property}) \end{aligned}$$

This functional is shown to have the following bounds:

Lemma 1 (*[8]*). *Let system (1) be exponentially stable. Given positive definite matrices* W_j, $j = 0, 1, 2$, *there exists* $\alpha_1 > 0$ *such that the complete type functional (2) admits the following quadratic lower bound:*

$$\alpha_1\left\|\varphi(0)\right\|^2 \leq V(\varphi), \quad \varphi \in \mathcal{PC}\left([-h, 0], \mathbb{R}^n\right).$$

Lemma 2 ([8]). *Let system (1) satisfy the Lyapunov condition [8]. Given symmetric matrices W_j, $j = 0, 1, 2$, for some positive α_2 functional (2) satisfy the inequality*

$$V(\varphi) \leq \alpha_2 \left\| \varphi \right\|_h^2, \quad \varphi \in \mathcal{PC}\left([-h, 0], \mathbb{R}^n\right).$$

The main ideas of the CLKF approach [6], are expressed in terms of complete type functionals. Consider time delay systems affine in the control input of the form

$$\dot{x}(t) = f(x_t) + g(x_t)u(t), \tag{3}$$

where

$$\begin{aligned} f(x_t) &= A_0 x(t) + A_1 x(t-h) + F(x(t), x(t-h)), \\ g(x_t) &= B(x(t), x(t-h)). \end{aligned} \tag{4}$$

Here, the smooth functionals $f: C \to \mathbb{R}^n$ and $g: C \to \mathbb{R}^{n \times m}$ are bounded on bounded sets with $f(0) = 0$, so that the system has zero solution when $u \equiv 0$. The control $u \in \mathbb{R}^m$ is a piecewise continuous function and the initial condition is given by a continuous vector valued function $x_0 = \varphi$, $\varphi: [-h, 0] \to \mathbb{R}^n$.

We present now the new definition of CLKF for complete type functionals:

Definition 1. A complete type functional $V(x_t): C \to \mathbb{R}^+$, whose explicit form is given by (3) is called a Control Lyapunov-Krasovskii Functional (CLKF) for system (3) and (4) if there exist a control law u and scalars α_1, α_2 such that

$$\alpha_1 \left\| \varphi(0) \right\|^2 \leq V(\varphi) \leq \alpha_2 \left\| \varphi \right\|_h^2, \quad \left. \frac{dV(x_t)}{dt} \right\|_{(3)-(4)} = \Psi_0(x_t) + \Psi_1^T(x_t)u < 0,$$

and

$$\Psi_1^T(x_t) = 0, \quad \textit{for} \quad x_t \neq 0 \Rightarrow \Psi_0(x_t) < 0,$$

for all piecewise continuous functions $\varphi: [-h, 0] \to \mathbb{R}^n$. Here

$$\Psi_0(x_t) = -x^T(t)W_0 x(t) - x^T(t-h)W_1 x(t-h) - \int_{-h}^{0} x^T(t+\theta)W_2 x(t+\theta)d\theta + 2\left[U(0)x(t) + \int_{-h}^{0} U(-h-\theta)A_1 x(t+\theta)d\theta\right]^T F(x(t), x(t-h)),$$

and

$$\Psi_1^T(x_t) = 2\left[U(0)x(t) + \int_{-h}^{0} U(-h-\theta)A_1 x(t+\theta)d\theta\right]^T B(x(t), x(t-h)).$$

2.2. Inverse optimality approach for delay free nonlinear systems

We introduce the definition of Control Lyapunov Function for systems evolving on $\mathbb{R}^n$ and affine in the control inputs:

$$\dot{x}(t) = f_0(x) + f_1(x)u, \tag{5}$$

where all entries of vector f_0 and the $n \times m$ matrix f_1 are smooth functions on $\mathbb{R}^n$, and $f(0) = 0$. We assume that the control inputs are restricted to take values in some subset of $\mathbb{R}^m$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$.

Definition 2 *([18]).* A proper and positive definite smooth function

$$\tilde{V}: \mathbb{R}^n \to \mathbb{R}^+$$

is said to be a Control Lyapunov Function (CLF) (with respect to control variables taking values in $\mathcal{U}$) if

$$\inf_{u \in \mathcal{U}} \left\{ L_{f_0}\tilde{V} + L_{f_1}\tilde{V}u \right\} < 0, \quad \forall x \neq 0,$$

where $L_{f_0}\tilde{V} = \left(\frac{\partial \tilde{V}}{\partial x}\right)^T f_0$, $L_{f_1}\tilde{V} = \left(\frac{\partial \tilde{V}}{\partial x}\right)^T f_1$.

The inverse optimality problem for delay free nonlinear affine systems of the form (5) is introduced in [1]. Assume that $\tilde{V}(x)$ is a Control Lyapunov Function for this system. This means that $\tilde{V}(x)$ is a positive definite, continuously differentiable function and there exists a control law u such that the derivative along the solutions of system (5) is

$$\left. \frac{d\tilde{V}(x)}{dt} \right|_{(5)} = \psi_0(x) + \psi_1^T(x)u < 0,$$

where $\psi_0 \in \mathbb{R}$,

$$\psi_0(x) = \nabla_x \tilde{V}(x) \cdot f_0(x)$$

and $\psi_1 \in \mathbb{R}^m$,

$$\psi_1(x) = \left[\nabla_x \tilde{V}(x)^T f_1(x)\right]^T.$$

As we assumed that $\tilde{V}(x)$ is a CLF for (5), then if $x \neq 0$ and $\psi_1(x)=0$ we have that $\psi_0(x)<0$ and the asymptotic stability of system (5) is guaranteed by suitable input u. We are then able to define the following positive definite scalar functions, which are well defined because they depend on the terms of the derivative of $\tilde{V}(x)$

$$q(x) \triangleq \psi_1^T(x)\psi_1(x) + \sqrt{\psi_0(x)^2 + \left[\psi_1^T(x)\psi_1(x)\right]^2},$$

$$r(x) \triangleq \frac{\frac{1}{4}\psi_1^T(x)\psi_1(x)}{d_r(x)},$$

where

$$d_r(x) = \psi_1^T(x)\psi_1(x) + \psi_0(x) + \sqrt{\psi_0(x)^2 + \left[\psi_1^T(x)\psi_1(x)\right]^2}.$$

Consider now the performance index:

$$\tilde{J} = \int_0^\infty f_m(x,u)dt, \tag{6}$$

where

$$f_m(x,u) = q(x) + r(x)u^T u$$

is a positive definite function. The Hamilton Jacobi Bellman (HJB) equation associated to system (5) and the performance index (6) is given by

$$\min_u \left(\left.\frac{d\tilde{V}(x)}{dt}\right|_{(5)} + f_m(x,u) \right) = 0. \tag{7}$$

It is easy to verify that the function $\tilde{V}(x)$ satisfies (7), hence it is a Bellman's function. Then, the optimal control

$$u^* = -\frac{1}{2}\frac{\psi_1(x^*)}{r(x^*)}$$

is obtained by differentiating equation (7) with respect to u [19].

Based on these ideas, we present our main results in the next section.

3. Problem formulation and main result

Consider a single state delay system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + F(x(t), x(t-h)) + B(x(t), x(t-h))u(t), \tag{8}$$

where the linear part of the system and initial condition are defined as in (1), and the input $u \subseteq \mathcal{U} \in \mathbb{R}^m$. We consider the following assumptions:

Assumption 1. The linear system is exponentially stable or, when matrix B is constant, there exists a preliminary stabilizing control law given by $\tilde{u} = K_0 x(t) + K_1 x(t-h) + u$.

Assumption 2. The matrix $B(x(t), x(t-h)) \in \mathbb{R}^{n \times m}$ is a continuous function with respect to its arguments.

Assumption 3. Let $\tilde{\omega}(x_t)$ a Lipschitz function, then $\left\|B(x(t), x(t-h)\tilde{\omega}(x_t))\right\| \le L_1 \|x_t\|_h$, with $L_1 > 0$, and $\|x_t\|_h < \delta_1$.

Assumption 4. The nonlinear part, described by the unknown function $F(x(t), x(t-h)) \in \mathbb{R}^n$, satisfies the condition:

$$\left\|F(x(t), x(t-h))\right\| \le \alpha \left\|x(t)\right\| + \beta \left\|x(t-h)\right\|, \tag{9}$$

where $\alpha, \beta \in \mathbb{R}^+$.

We want to synthesize a nonlinear optimal control law $u(t)$ such that the closed loop system is stable. To do so, we extend the inverse optimality approach for delay free systems [1] to this particular case of nonlinear time delay systems. We first find the sufficient condition guaranteeing that the complete type functional (2) is a CLKF for system(8). Then, by taking the CLKF as a Bellman functional and considering the dynamic programming approach with a specific performance index we are able to find the optimal nonlinear controller without solving the HJB type equation. The time derivative of the functional (2) along the trajectories of system (8) is

$$\begin{aligned}\left.\frac{dV(x_t)}{dt}\right|_{(8)} &= -x^T(t)Wx(t) + 2x^T(t)U(0)[F(x(t), x(t-h)) + B(x(t), x(t-h))u(t)] \\ &\quad +2[F(x(t), x(t-h)) + B(x(t), x(t-h))u(t)]^T \int_{-h}^{0} U(-h-\theta)A_1x(t+\theta)d\theta + x^T(t)[W_1 + hW_2]x(t) - x^T(t-h)W_1x(t-h) \\ &\quad - \int_{-h}^{0} x^T(t+\theta)W_2x(t+\theta)d\theta, \\ &= -\omega_0(x_t) + 2[F(x(t), x(t-h)) + B(x(t), x(t-h))u(t)]^T\omega_1(x_t),\end{aligned}$$

where

$$\omega_0(x_t) = x^T(t)W_0x(t) + x^T(t-h)W_1x(t-h) + \int_{-h}^{0} x^T(t+\theta)W_2x(t+\theta)d\theta \tag{10}$$

and

$$\omega_1(x_t) = U(0)x(t) + \int_{-h}^{0} U(-h-\theta)A_1x(t+\theta)d\theta. \tag{11}$$

This can be rewritten as

$$\left.\frac{dV(x_t)}{dt}\right|_{(8)} = \Psi_0(x_t) + \Psi_1^T(x_t)u(t), \tag{12}$$

where $\Psi_0 \in \mathbb{R}$,

$$\Psi_0(x_t) = -\omega_0(x_t) + 2\omega_1^T(x_t)F(x(t), x(t-h)) \tag{13}$$

and $\Psi_1 \in \mathbb{R}^m$,

$$\Psi_1^T(x_t) = 2\omega_1^T(x_t)B(x(t), x(t-h)). \tag{14}$$

Observe that, as in [11] if the functional $\Psi_1^T(x_t) \neq 0$ for all $x_t \neq 0$, a nonlinear control law can be applied. According to Definition 1, when the functional $\Psi_1^T(x_t) = 0$ but $x_t \neq 0$, which means that the system response has not converged to the origin, the sufficient condition that guarantees that the system (8) is asymptotically stable is that the term $\Psi_0(x_t)$ is negative, hence in the next section we verify the conditions under which the inequality

$$\left.\frac{dV(x_t)}{dt}\right|_{(8)} = \Psi_0(x_t) = -\omega_0(x_t) + 2\omega_1^T(x_t)F(x(t), x(t-h)) < 0. \tag{15}$$

is satisfied. Substituting (10) and (11) into the previous expression, we obtain

$$\begin{aligned}\left.\frac{dV(x_t)}{dt}\right|_{(8)} &= -x^T(t)W_0x(t) - x^T(t-h)W_1x(t-h) - \int_{-h}^{0} x^T(t+\theta)W_2x(t+\theta)d\theta \\ &\quad +2\left[U(0)x(t) + \int_{-h}^{0} U(-h-\theta)A_1x(t+\theta)d\theta\right]^T F(x(t), x(t-h)).\end{aligned} \tag{16}$$

3.1. Main result

In this section, the condition under which the functional (2) is a CLKF for system (8) is presented. The result is then employed in the design of the nonlinear control law. Moreover, it is proved that the performance index is bounded.

3.1.1. Sufficient conditions for $V(x_t)$ to be a CLKF

The following proposition provides sufficient conditions under which a complete type functional is a CLKF.

Proposition 1. *Let the nonlinear time delay system (8) and let positive definite matrices $W_j \in R^{n\times n}$, $j = 0, 1, 2$, $W = W_0 + W_1 + hW_2$ be given. If there exists a scalar $\epsilon > 0$ such that the matrix*

$$E = \begin{bmatrix} W_0 - \epsilon\bar{\alpha}I_n & 0_{n\times n} & 0_{n\times n} & -U(0) \\ 0_{n\times n} & W_1 - \epsilon\bar{\beta}I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \frac{1}{h}\frac{W_2}{\bar{\varsigma}^2} & -I_n \\ -U(0) & 0_{n\times n} & -I_n & \epsilon I_n \end{bmatrix} \tag{17}$$

is positive definite, then the complete type functional $V(x_t)$ given by (2) is a Control Lyapunov-Krasovskii Functional for system (8).

Proof 1. Consider equation (16), in order to majorize the quadratic term in W_2, in (16) we proceed as follows: As the nominal linear system (1) is assumed to be asymptotically stable, $U(0)$ is a positive definite matrix [20]. This in turn implies that the term $S(\theta)=U(-h-\theta)A_1 \in \mathbb{R}^{n\times n}$ is not equal to the zero function for all $\theta \in [-h, 0]$. Let us define

$$\bar{s} = \sup_{\theta\in[-h,0]} \|S(\theta)\|. \tag{18}$$

As

$$S^T(\theta)S(\theta) \le \left\|S(\theta)\right\|^2$$

we obtain that

$$\frac{S^T(\theta)S(\theta)}{\left\|S(\theta)\right\|^2} \le I_n,$$

and

$$\frac{1}{\bar{s}} \le \frac{1}{\sup_{\theta\in[-h,0]}\left\|S(\theta)\right\|} \le \frac{1}{\left\|S(\theta)\right\|}, \quad \theta \in [-h, 0].$$

Then, the term in W_2 admits the majorization

$$\begin{aligned}\int_{-h}^{0} x^T(t+\theta)W_2x(t+\theta)d\theta &\ge \int_{-h}^{0} x^T(t+\theta)\left(\frac{S(\theta)}{\left\|S(\theta)\right\|}\right)^T W_2 \left(\frac{S(\theta)}{\left\|S(\theta)\right\|}\right) x(t+\theta)d\theta \\ &\ge \int_{-h}^{0} x^T(t+\theta)S^T(\theta)\frac{W_2}{\bar{s}^2}S(\theta)x(t+\theta)d\theta,\end{aligned}$$

and the Jensen inequality implies that

$$-\int_{-h}^{0} x^T(t+\theta)W_2x(t+\theta)d\theta \le -\frac{1}{h}\left(\int_{-h}^{0} S(\theta)x(t+\theta)d\theta\right)^T \frac{W_2}{\bar{s}^2}\left(\int_{-h}^{0} S(\theta)x(t+\theta)d\theta\right). \tag{19}$$

Replacing (19) into (16), we get

$$\begin{aligned}\left.\frac{dV(x_t)}{dt}\right|_{(8)} &\le -x^T(t)W_0x(t) - x^T(t-h)W_1x(t-h) - \frac{1}{h}\left(\int_{-h}^{0} S(\theta)x(t+\theta)d\theta\right)^T \frac{W_2}{\bar{s}^2}\left(\int_{-h}^{0} S(\theta)x(t+\theta)d\theta\right) \\ &\quad + 2x^T(t)U(0)F(x(t), x(t-h)) + 2\left[\int_{-h}^{0} S(\theta)x(t+\theta)d\theta\right]^T F(x(t), x(t-h)).\end{aligned}$$

Applying the S-procedure [21] as in [22], we add and subtract the term

$$\pm\epsilon F^T(x(t), x(t-h))F(x(t), x(t-h)), \quad \epsilon > 0.$$

We obtain

$$\left.\frac{dV(x_t)}{dt}\right|_{(8)} \le -\eta^T \begin{bmatrix} W_0 & 0_{n\times n} & 0_{n\times n} & -U(0) \\ 0_{n\times n} & W_1 & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \frac{1}{h}\frac{W_2}{\bar{s}^2} & -I_n \\ -U(0) & 0_{n\times n} & -I_n & \epsilon I_n \end{bmatrix} \eta + \epsilon F^T(x(t), x(t-h))F(x(t), x(t-h)),$$

where I_n is the identity matrix of n dimension, $0_{n\times n}$ is a $n \times n$ block of zeros,

$$\eta^T = \left[x(t), \quad x(t-h), \quad \int_{-h}^{0} S(\theta)x(t+\theta)d\theta, \quad F(x(t), x(t-h))\right]$$

and from the assumption that the nonlinear function $F(x(t), x(t-h))$ satisfies (9) we get

$$\epsilon F^T(x(t), x(t-h))F(x(t), x(t-h)) \le \epsilon\bar{\alpha}x^T(t)x(t) + \epsilon\bar{\beta}x(t-h)^Tx(t-h),$$

for some $\bar{\alpha}, \bar{\beta} \in \mathbb{R}^+$. Using this inequality, we arrive at

$$\left.\frac{dV(x_t)}{dt}\right|_{(8)} \le -\eta^T E\eta,$$

with matrix E given by (17) and the proposition is proved.□

Remark 1. The matrix parameters W_i, $i = 0, 1, 2$, and ϵ could be found by considering (17) as a Linear Matrix Inequality (LMI) ($E > 0$), under the restriction $W = W_0 + W_1 + hW_2$.

3.1.2. Synthesis of the optimal nonlinear control law via inverse optimality approach

In view of Proposition 1, we can now extend the inverse optimality approach for delay free nonlinear systems proposed in [1] to time delay systems of the form (8).

Proposition 2. *Suppose that the functional $V(x_t)$ given by (2) satisfies the condition established in Proposition 1, then the optimal control law*

$$u^*(t) = \begin{cases} -\dfrac{1}{2}\dfrac{\Psi_1(x_t^*)}{r(x_t^*)}, & \Psi_1(x_t^*) \neq 0 \\ 0, & \Psi_1(x_t^*) = 0, \text{ or } x_t^* = 0 \end{cases}, \tag{20}$$

stabilizes the system (8) in the local sense and minimizes the performance index (21).

Proof 2. Assume that functional $V(x_t)$ is a CLKF of the system (8), and consider the following performance index:

$$J = \int_0^\infty \left[q(x_t) + r(x_t)u^T u\right] dt, \tag{21}$$

where $q(x_t)$ and $r(x_t)$ are defined as follows:

$$q(x_t) = \left[\Psi_1^T(x_t)\Psi_1(x_t)\right] + \sqrt{[\Psi_0(x_t)]^2 + \left[\Psi_1^T(x_t)\Psi_1(x_t)\right]^2}, \tag{22}$$

$$r(x_t) = \frac{\frac{1}{4}\left[\Psi_1^T(x_t)\Psi_1(x_t)\right]}{d_r(x_t)},$$

$$d_r(x_t) = \Psi_1^T(x_t)\Psi_1(x_t) + \Psi_0(x_t) + \sqrt{[\Psi_0(x_t)]^2 + \left[\Psi_1^T(x_t)\Psi_1(x_t)\right]^2},$$

with $\Psi_0(x_t)$ and $\Psi_1(x_t)$ are given by equations (13) and (14). The functionals $q(x_t)$ and $r(x_t)$ are strictly positive definite and these depend on the derivative of $V(x_t)$ therefore they are well defined. In fact, it is not hard to see that the functional $q(x_t)$ is definite positive when $\Psi_1(x_t) \neq 0$. Now, for the functional $r(x_t)$, it is clear that $d_r(x_t)$ is positive definite when $\Psi_1(x_t) \neq 0$: observe that $\Psi_0(x_t) < 0$ and we can suppose that $d_r(x_t) \leq 0$, which implies that $\Psi_1^T(x_t)\Psi_1(x_t) + \sqrt{[\Psi_0(x_t)]^2 + \left[\Psi_1^T(x_t)\Psi_1(x_t)\right]^2} \leq -\Psi_0(x_t)$, $(-\Psi_0(x_t) > 0)$. Squaring on both sides, we arrive at

$$\left[\Psi_1^T(x_t)\Psi_1(x_t)\right]^2 + \Psi_1^T(x_t)\Psi_1(x_t)\sqrt{[\Psi_0(x_t)]^2 + \left[\Psi_1^T(x_t)\Psi_1(x_t)\right]^2} \leq 0,$$

which is not possible. Therefore $d_r(x_t) > 0$ when $\Psi_0(x_t) < 0$ and $\Psi_1(x_t) \neq 0$.

Next, the HJB equation associated to system (8) and to the performance index J is given by

$$\min_u \left(\left.\frac{dV(x_t)}{dt}\right|_{(8)} + q(x_t) + r(x_t)u^T u \right) = 0. \tag{23}$$

Replacing (12) into (23) and computing the first derivative with respect to u, we get the control law

$$u^* = -\frac{1}{2}\frac{\Psi_1(x_t^*)}{r(x_t^*)}.$$

As the second derivative of (23) with respect to u^* is equal to $2r(x_t^*) > 0$, we are in presence of a minimum, hence the control law is optimal. The stability of the closed loop is ensured because $V(x_t^*)$ is a CLKF and the HJB equation is satisfied with the control law u^* (see Appendix). Nevertheless, a singular point can appear in the control law when the function $\Psi_1(x_t^*) = 0$, for some $x_t^* \neq 0$, which makes that $r(x_t^*)$ is undefined. This singularity in the control law must be removed making $u^* = 0$ when $\Psi_1(x_t^*) = 0$. □

Remark 2. The control law (20) is given by

$$u^*(t) = 2\left(\frac{\Psi_1\left(x_t^*\right)\left(\Psi_1^T\left(x_t^*\right)\Psi_1\left(x_t^*\right) + \Psi_0\left(x_t^*\right) + \sqrt{\Psi_0^2\left(x_t^*\right) + \left[\Psi_1^T\left(x_t^*\right)\Psi_1\left(x_t^*\right)\right]^2}\right)}{\Psi_1^T\left(x_t^*\right)\Psi_1\left(x_t^*\right)}\right), \tag{24}$$

where $\Psi_0(x_t^*)$ and $\Psi_1^T(x_t^*) \neq 0$, are given by (13) and (14), respectively. Notice that Assumption 3 implies that functional $\Psi_1^T(x_t^*)$ is locally Lispchitz i.e., it satisfies $\left\|\Psi_1(x_t^*)\right\| \leq \alpha_0\left\|x_t^*\right\|_h$, for some $\alpha_0 > 0$, and $\left\|x_t^*\right\|_h < \delta$. The term $\Psi_0\left(x_t^*\right)$ satisfies $\left|\Psi_0(x_t^*)\right| \leq \alpha_1\left\|x_t^*\right\|_h^2$, where $\alpha_1 = \left\|W_0\right\| + \left\|W_1\right\| + h\left\|W_2\right\| + \left[\max_{\theta\in[0,h]}\left\|U(\theta)\right\|\left(1 + h\left\|A_1\right\|\right)\right] L_1$, and L_1 is obtained as: $\left\|F(x^*(t), x^*(t-h))\right\| \leq \alpha\left\|x^*(t)\right\| + \beta\left\|x^*(t-h)\right\| \leq L_1\left\|x_t^*\right\|_h$, $L_1 = \alpha + \beta$, $\left\|x_t^*\right\|_h < \delta$. Then, the norm of the numerator of the control law (24) satisfies

$$\left\|\Psi_1\left(x_t^*\right)\left(\Psi_1^T\left(x_t^*\right)\Psi_1\left(x_t^*\right) + \Psi_0\left(x_t^*\right) + \sqrt{\Psi_0^2\left(x_t^*\right) + \left[\Psi_1^T\left(x_t^*\right)\Psi_1\left(x_t^*\right)\right]^2}\right)\right\| \leq \alpha_2\left\|x_t^*\right\|_h^3,$$

where $\alpha_2 = 4\left(\alpha_0^3 + \alpha_0\alpha_1\right)$ and $\left|\Psi_1^T\left(x_t^*\right)\Psi_1\left(x_t^*\right)\right| \le \alpha_0^2\left\|x_t^*\right\|_h^2$, as the term in the numerator has a higher degree than the denominator, the convergency rate of the numerator is greater, therefore when $\left\|x_t^*\right\|_h \to 0$, $u^*(t) \to 0$, and for all $\tilde{\varepsilon} > 0$, there exists $\delta > 0$ such that $\left\|x_t^*\right\|_h < \delta(\tilde{\varepsilon}) \Rightarrow \left\|u^*(t)\right\| < \tilde{\varepsilon}$. This implies that the complete type functional $V(x_t^*)$ satisfies the small control property [6], and we conclude that the control law u^* is continuous at $x_t^* = 0$.

3.1.3. Proof of convergency of the integral of the performance index

In order to verify that the integral (21) converges, we present the following analysis which demonstrates that the performance index is bounded. Consider the system (8) and start from the assumption that the functional $V(x_t)$ given by (2) is a CLKF. The HJB equation implies that

$$\left.\frac{dV(x_t^*)}{dt}\right|_{(8),u=u^*} = -q(x_t^*) - r(x_t^*)u^{*T}u^*,$$

where x_t^* is the optimal trajectory generated by u^*. Now, integrating from 0 to ∞ we get

$$\lim_{t\to\infty} V(x_t^*) - V(x_0) = -\int_0^\infty \left(q(x_t^*) + r(x_t^*)u^{*T}u^*\right) dt.$$

The fact that $V(x_t)$ is a CLKF guarantees the asymptotic stability of system (8), hence

$$\lim_{t\to\infty} V(x_t^*) = 0,$$

and we obtain

$$V(\varphi) = \int_0^\infty \left(q(x_t^*) + r(x_t^*)u^{*T}u^*\right) dt.$$

Observe that this is the optimal value for the performance index given by (21), which is the functional $V(x_t)$ evaluated at the initial condition, hence

$$\left|V(\varphi)\right| = V(\varphi) \le M,$$

where M is a positive scalar. We conclude that (21) is bounded by M.

4. Inverse optimal control of a dehydration prototype

In this section, experimental results on a dehydration process are carried out, thus extending the simulation results in [23]. The experimental platform and its mathematical model are introduced first. Stabilization tests, response to disturbances, and energy savings achieved by our optimal design are contrasted with an industrial PID controller.

4.1. Description of the dehydration prototype

It is worth mentioning that our objective is to reach and to maintain a desired constant temperature setpoint, and to test the performance given by the control law (20), which minimizes the index (21). The prototype emulates an industrial atmospheric dehydrator and has a similar design as the one presented in [24]. It consists of a closed box with a wind tunnel as output and a tube that recycles the hot air into the system. The main elements of the prototype are:

- An electrical grid used as the source of heat (the actuator). The temperature in the box is regulated by the voltage that is applied. This voltage is our control input.
- A low power stage: The output value of the control law is converted to a Pulse-Width Modulation (PWM) signal, in a rank of 0–5 volts, and then it is sent to a high power stage.
- A high power stage: The PWM signal from the low power stage is applied to a set of electronic circuits. First the AC voltage is converted to DC voltage and then according to the PWM signal the control is applied to the electrical grid. Considering a liner relationship, when the low power stage provides 5 V in the high power stage we obtain 180 V.
- The process variable (temperature) measured by using the temperature sensor LM35 with a rate of 10 mV/°C
- A fan producing a constant air flow of 2.1 m/s. This value was chosen according to recommendations in the specialized literature [25].
- A Data Acquisition Card (DAQ USB-6008) and software LabView of National Instruments used to implement the control law and store the process data. The sample time is 0.5 s.
- An industrial PID Honeywell DC1040 with maximum precision of ±1 °C of cold junction compensation, ±5% of maximum deviation in the linear output 4–20 mA, automatic compensation of dead zone, and with a thermocouple J type of extended rate in the input.

Fig. 1 shows the diagram of the prototype. The air flow passes through the electrical grid where it is heated, then it reaches the plate where the product (some slices of tomatoes) and the temperature sensor are placed. Part of the air goes through the wind tunnel and the other part is recycled. The temperature value is measured and used to compute the control voltage.

The hot air recycling loop induces a state delay, therefore the process is modeled as

$$\dot{x}(t) = a_0x(t) + a_1x(t-h) + bu(t) + f(x(t), x(t-h)), \tag{25}$$

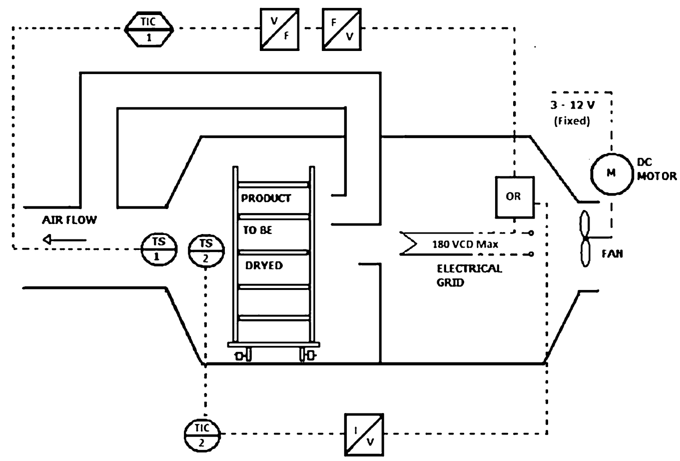

Fig. 1. Diagram of the dehydration prototype.

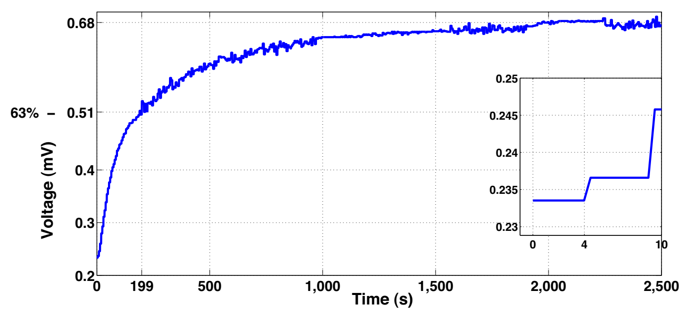

Fig. 2. Step response of the dehydration process.

where the temperature value is the state variable $x(t)$, the control input $u(t)$ is the energy applied to the actuator. Following the discussion in [12], the nonlinearity is assumed to be a polynomial function depending on the process variable

$$f(x(t), x(t-h)) = cx^2(t) + dx^2(t-h) + ex^3(t) + kx^3(t-h). \tag{26}$$

The time delay h is found heuristically to be of 10 s. All parameters are estimated by using a least square recursive method for different operation regions. The Lyapunov matrix $U(\theta), \theta \in [0, 10]$ is constructed by using the semi-analytical method [8]. The control law is computed according to (20) and the integral terms of (10) and (11) are approximated by using the Simpsonś rule [26].

It is worthy of mention that despite the fact that system (21) is controllable, experimental results carried out with feedback exact linearization control law gave a poor performance because the error between the nominal and the approximated model cannot be eliminated. In fact, consider the nominal system $\dot{x}(t) = a_0 x(t) + g(x_t) + bu(t)$, where $g(x_t) = a_1 x(t-h) + f(x(t), x(t-h))$, and the feedback linearization control law $u(t) = u_2 - \frac{1}{\tilde{b}}\tilde{g}(x_t)$, here $\tilde{g}(x_t)$ is the polynomial approximation of the nominal value $g(x_t)$ [12], and assume that $a_0 \approx \tilde{a}_0$, $b \approx \tilde{b}$, where $\tilde{a}_0$ and $\tilde{b}$ are the estimated parameters of the nominal parameters a_0 and b. The closed loop system is $\dot{x}(t) = a_0 x(t) + bu_2 + \Delta(x_t)$ where $\Delta(x_t) = g(x_t) - \tilde{g}(x_t)$, and $\left|\Delta(x_t)\right| \leq m,\ m > 0$, is the error between the nominal and the approximated systems. Although this error is bounded in a region (Weierstrass Approximation Theorem [27]), and the solutions of the nominal system and the perturbed system are exponentially bounded (continuity properties [8]), for this process, the error affects the system response, and as a consequence, the product quality.

Now, we want to test the optimal nonlinear control performance by comparing it against an industrial PID controller Honeywell DC1040. Notice that, to get a functioning controller, one must consider filtering of the measured signal, protection for integral windup, as well as bumpless mode and parameter changes [28]. Moreover, in process control more than 95% of the control loops are of PID type, and 20% of the loops use "factory tuning", i.e., operate with default parameters set by the controller manufacturer [29]. Here, the PID controller is tuned according to the Ziegler-Nichols tuning rules [30] based on the step response of the plant which is shown in Fig. 2.

The dehydration process is represented by the transfer function

$$\frac{X(s)}{U(s)} = \frac{Ke^{-hs}}{Ts + 1}, \tag{27}$$

where $T = 195$ s, $h = 4$ s and $K = 0.68$, and the obtained gains for the PID are $K_p = 58.5$, $K_i = 7.3$ and $K_d = 117$. This transfer function is a first order linear system with input delay, it can be viewed as a simplified model of a thermal process [31]. The parameters h, T and K were obtained from Fig. 2, by using the well know step response method [30,31]. Linear models (first and second order) with input delay are currently used in the industry by the software Expert-Tune [32] on its product PIDLoop Optimizer, which uses algorithms for a PID robustly tuned.

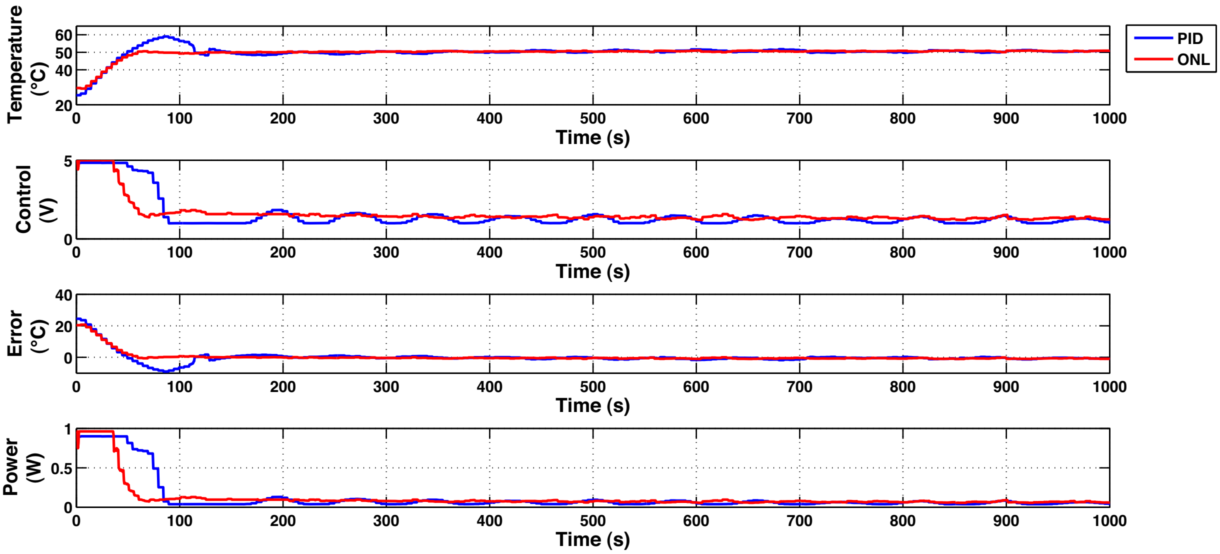

Fig. 3. Stabilizing test, setpoint 50 °C.

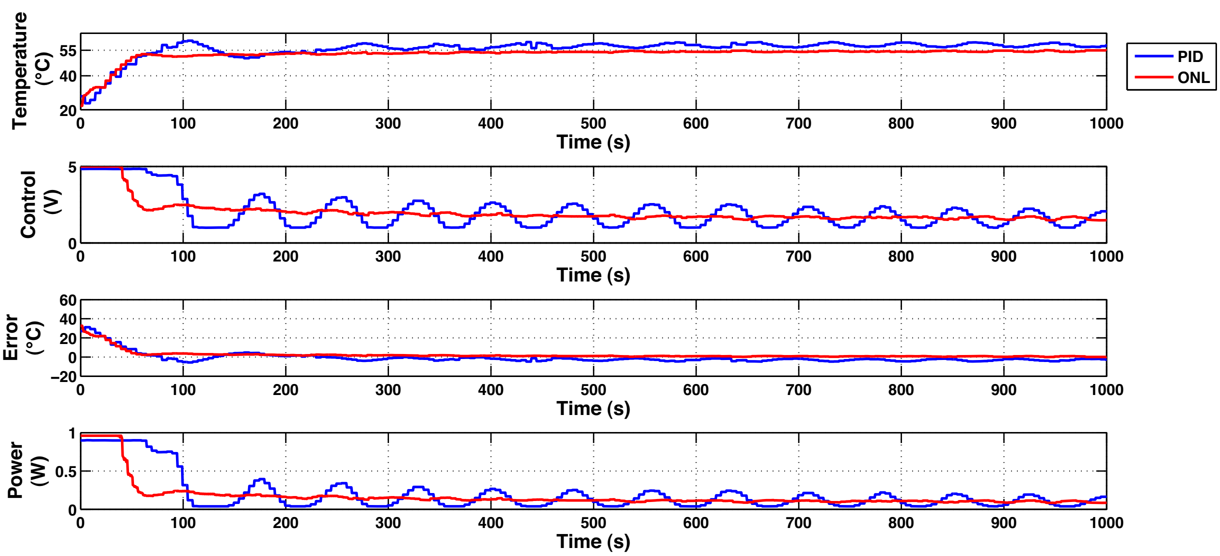

Fig. 4. Stabilizing test, setpoint 55 °C.

4.2. Stabilizing experimental results

The first experiment consists in stabilizing the temperature at 50 °C. The estimated parameters for this setpoint are: $a_0 = -0.1346$, $a_1 = 0.0513$, $b = 0.0513$, $c = 0.0457$, $d = -0.5402$, $e = 0.21007$ and $k = -1.0467$. We consider the values: $W_0 = 200$, $W_1 = W_2 = 100$, $W = 1300$, $U(0) = 5854.630$, the scalars $\bar{s} = 300.875$, $\bar{\alpha} = \bar{\beta} = 0.017$ and $\epsilon = 346020$. The matrix (17) has the following eigenvalues $\{346119.0601,\ 0.9397,\ 0.00041,\ 0.00022\}$. The variables depicted on Figs. 3–10 are the temperature, the control voltage, the temperature error, and the power consumption in the low power stage. The constants given in the Remark 2 are $\alpha_0 = 1 \times 10^6$, $\tilde{\alpha}_0 = 9 \times 10^5$ and $\alpha_1 = 1601.3$.

The estimated parameters for a setpoint of 55 °C are: $a_0 = -0.0719$, $a_1 = -0.0441$, $b = 5.446 \times 10^{-4}$, $c = 0.0135$, $d = 0.3968$, $e = -0.0795$ and $k = -0.3637$. The gains are: $W_0 = 500$, $W_1 = W_2 = 100$, $W = 1600$, $U(0) = 9502.363$, the scalars $\bar{\alpha} = \bar{\beta} = 0.015$, $\epsilon = 444440$ and $\bar{s} = 419.994$. The eigenvalues of matrix (17) are $\{444809.452,\ 30.548,\ 0.000038,\ 0.001\}$. The system response is shown in Fig. 4.

For a setpoint of 60 °C the parameters are: $a_0 = -3.1345$, $a_1 = 0.6312$, $b = 0.0017$, $c = 9.9308$, $d = -3.8446$, $e = -8.3475$ and $k = 3.8023$. We consider the values: $W_0 = 300$, $W_1 = 200$, $W_2 = 100$, $W = 1500$, $U(0) = 244.271$, the scalars $\bar{\alpha} = \bar{\beta} = 0.5$, $\epsilon = 790$ and $\bar{s} = 154.193$. The matrix (17) has the following eigenvalues $\{867.9511,\ 24.5440,\ 0.0052,\ 2.5\}$ and the system response is given in Fig. 5.

Fig. 6 shows the experimental results for a setpoint of 70 °C. The estimated parameters are: $a_0 = -3.5102$, $a_1 = 0.6139$, $b = 8.7524 \times 10^{-4}$, $c = 7.8744$, $d = -2.3229$, $e = -5.2335$ and $k = 1.8404$ and the gains are: $W_0 = 600$, $W_1 = 500$, $W_2 = 100$, $W = 2100$, $U(0) = 303.7651$, the scalars $\bar{\alpha} = \bar{\beta} = 0.8$, $\epsilon = 740$, $\bar{s} = 186.508$ and the eigenvalues of (17) are $\{864.939,\ 1.3528,\ 0.1083,\ 0.2639\}$.

In Table 1, we present a comparison in terms of power consumption for these experiments. We compute the average power for the high power stage as $P_a = \frac{1}{T}\int_0^T P(t)dt$, where $P(t)$ is the instantaneous power and T is the duration of the experiment. Alternatively, to illustrate this result, we present an economic savings for industrial applications. The average retail price of electricity to ultimate industry customers

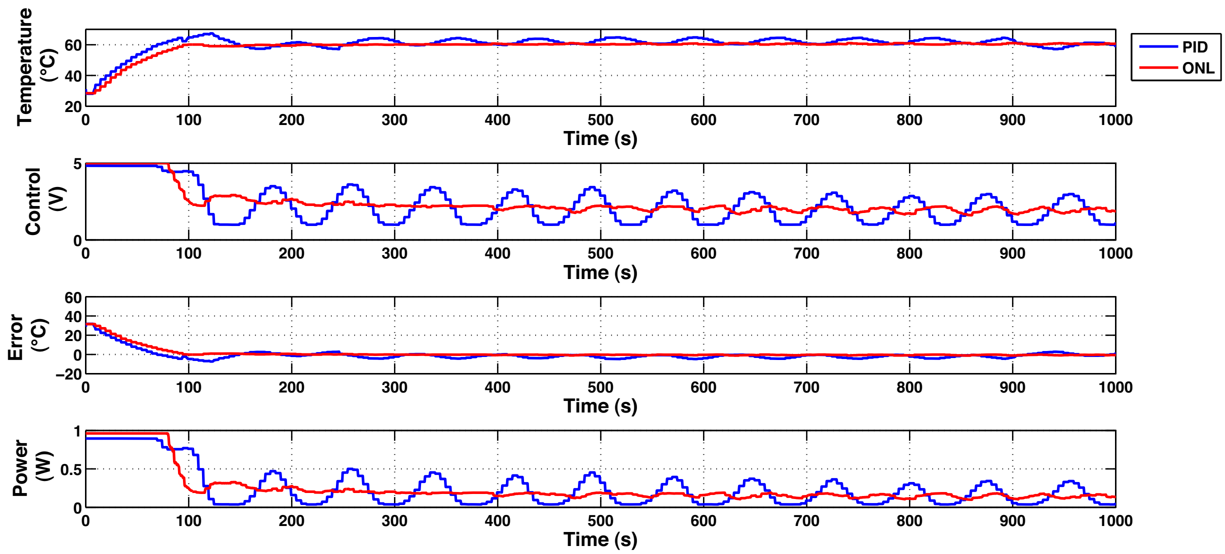

Fig. 5. Stabilizing test, setpoint 60 °C.

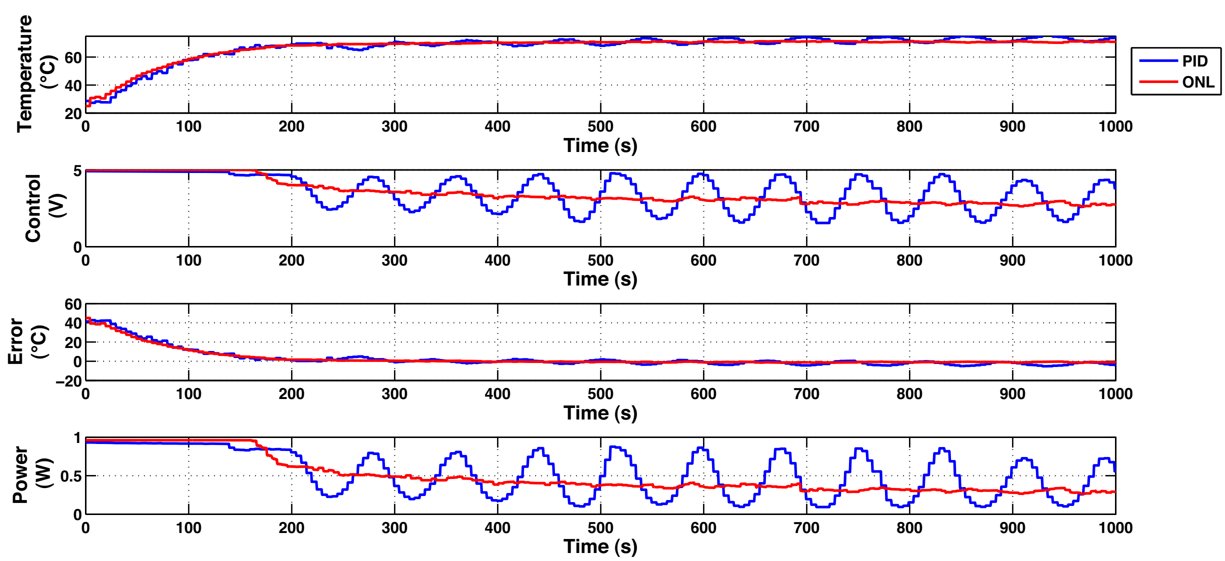

Fig. 6. Stabilizing test, setpoint 70 °C.

Table 1
Average power of the stabilizing tests.

Temperature(°C)	ONL (Wh)	PID (Wh)	Power savings (%)	Savings per year (US-dollar)
50	28.841	29.652	2.73	515.79
55	42.075	49.060	14.23	4451.01
60	61.033	62.184	1.85	733.45
70	121.856	132.587	8.09	6838.10

in November of 2014 for the "Pacific Contiguous" census division is 8.51 Cents per Kilowatt-hour [33]. Table 1 presents the total energy saving using one atmospheric drying process for industrial proposes (8-hours per day and 5-days per week).

Remark 3. Observe in Table 1 that the energy saved in the experiments at 50 and 60 degrees is low. However, for an error criteria of ±5%, when the optimal nonlinear control is applied, the settling time is 60 s and 100 s, respectively. For the PID, the settling time for these setpoints is 130 s and 140 s, respectively.

Remark 4. In view of the obtained constant values $\bar{\alpha}$ and $\bar{\beta}$ which satisfy the positivity of the matrix given by (17), we observe that in the case of a temperature setpoint of 50 °C the nonlinear function (26) satisfies the condition

$$f^2(x(t), x(t-h)) \leq 0.017\left(x^2(t) + x^2(t-h)\right).$$

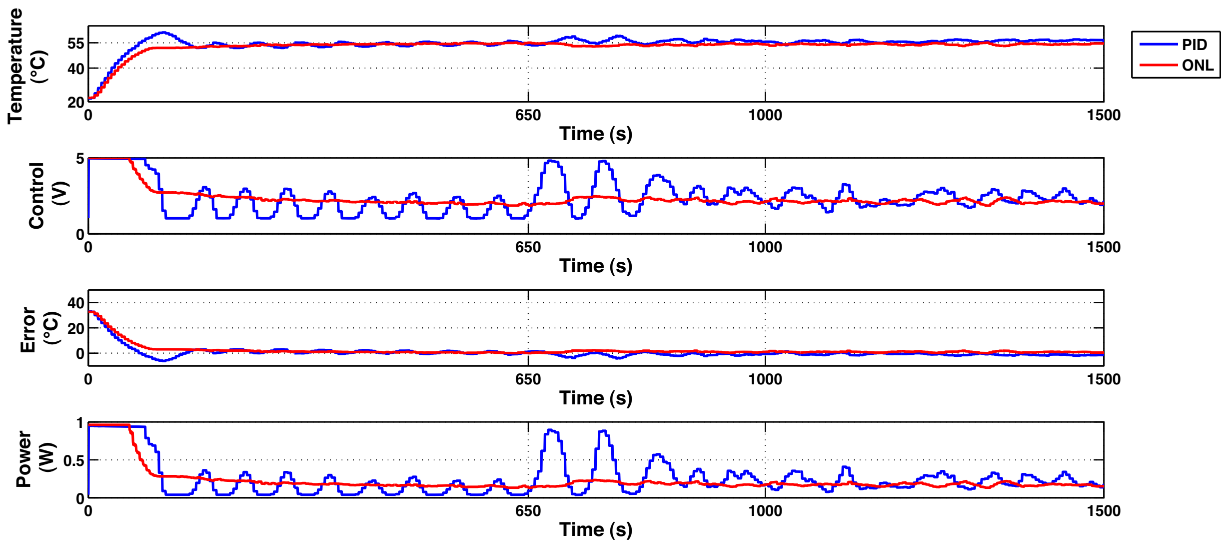

Fig. 7. Robustness test, setpoint 55 °C.

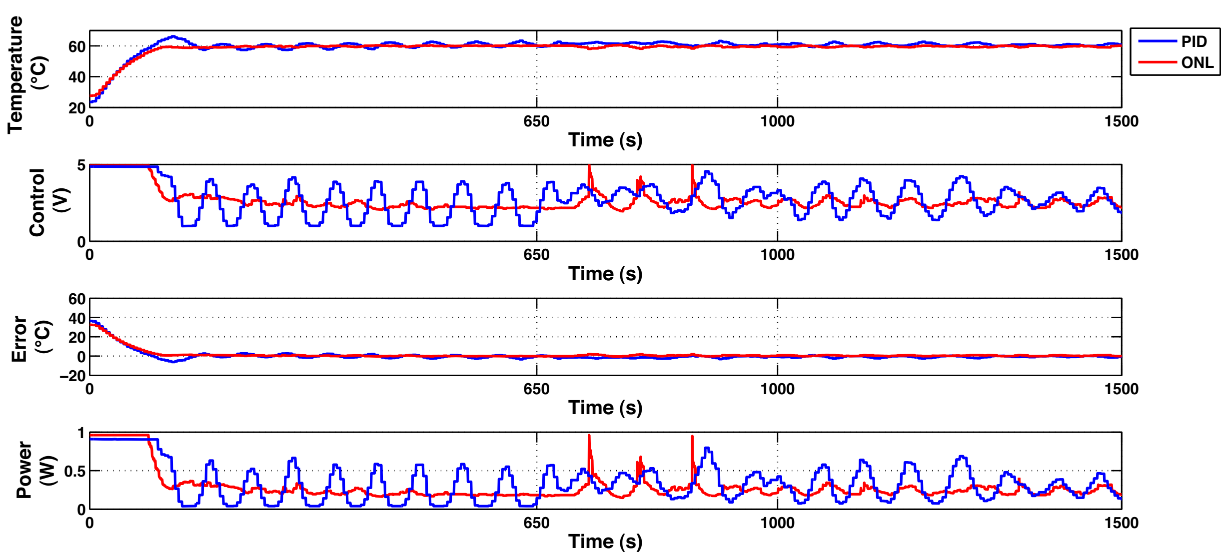

Fig. 8. Robustness test, setpoint 60 °C.

When the temperature reaches the steady state we find that the bound on the nonlinearities for which the temperature system can be maintained is ±9.22 °C; for a temperature setpoint of 55 °C the nonlinearities are ±9.53 °C. In contrast, when the setpoint is 60 °C we find a bound on the nonlinearities of ±30 °C and for 70 °C of ±560 °C. These very different bounds depend on the estimated parameters of the system, so we present some experiments to verify the robustness directly on the platform.

4.3. Robustness experimental results

In a dehydration process, the humidity of the product is measured, by weighing the product at given time intervals. This offline task requires opening the lid of the prototype, hence a disturbance is introduced. In this section, we present the experimental results showing the systems response to opening the lid 650 s after the beginning of the experiments. The gains considered in the previous subsection are used. Fig. 7 illustrates the system response for a setpoint of 55 °C.

Fig. 8 shows the experimental results when the setpoint is 60 °C. Then, Fig. 9 illustrates the results when the setpoint is 70 °C and the lid is opened 750 s after the beginning of the experiment.

Next in Table 2, we present a comparison of the power consumption for the robustness tests.

Remark 5. The robustness experiments show the efficiency of the proposed control law, give evidence of a better performance and more power savings, when we compare with an industrial PID controller.

Remark 6. Previous experimental results over temperature process were presented recently [34,35], where the optimal regulation of air flow in a dehydration problem with a prototype of a dryer is considered. In order to compare the performance of our controller (20) against a linear optimal controller with delay compensation, we consider a predictive control for systems with input delay as in [36,37] for compensating the time delay. Then, the optimal controller is synthesized as a Linear Quadratic Regulator [19].

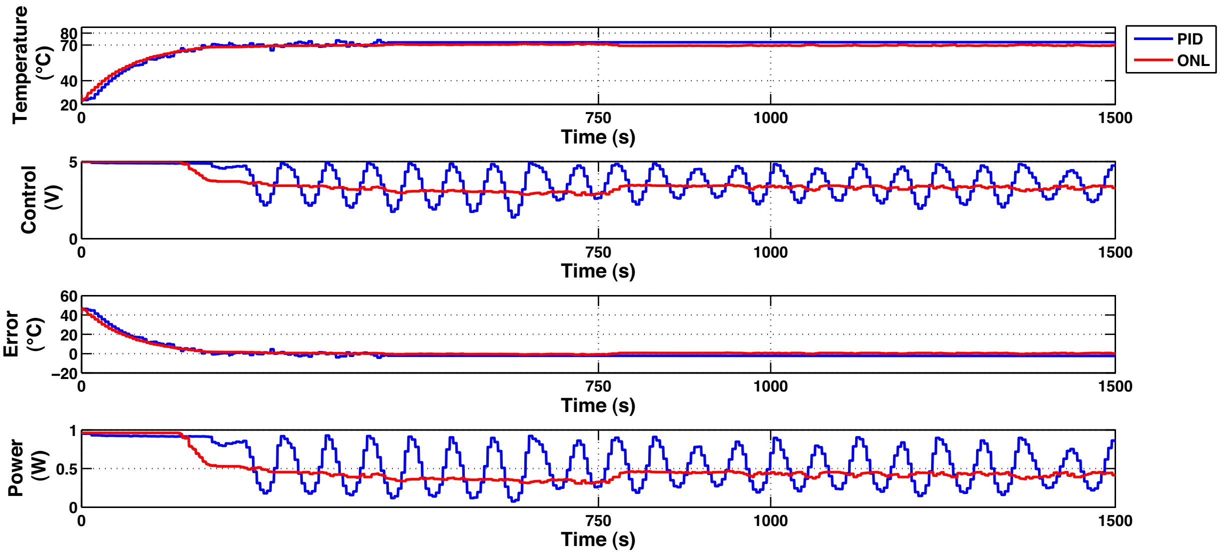

Fig. 9. Robustness test, setpoint 70 °C.

Table 2
Average power of the robustness tests.

Temperature (°C)	ONL (Wh)	PID (Wh)	Power savings (%)	Saving per year (US-dollar)
55	83.607	96.192	13.08	7979.37
60	107.768	125.919	14.41	11,566.39
70	178.535	213.673	16.44	22,390.94

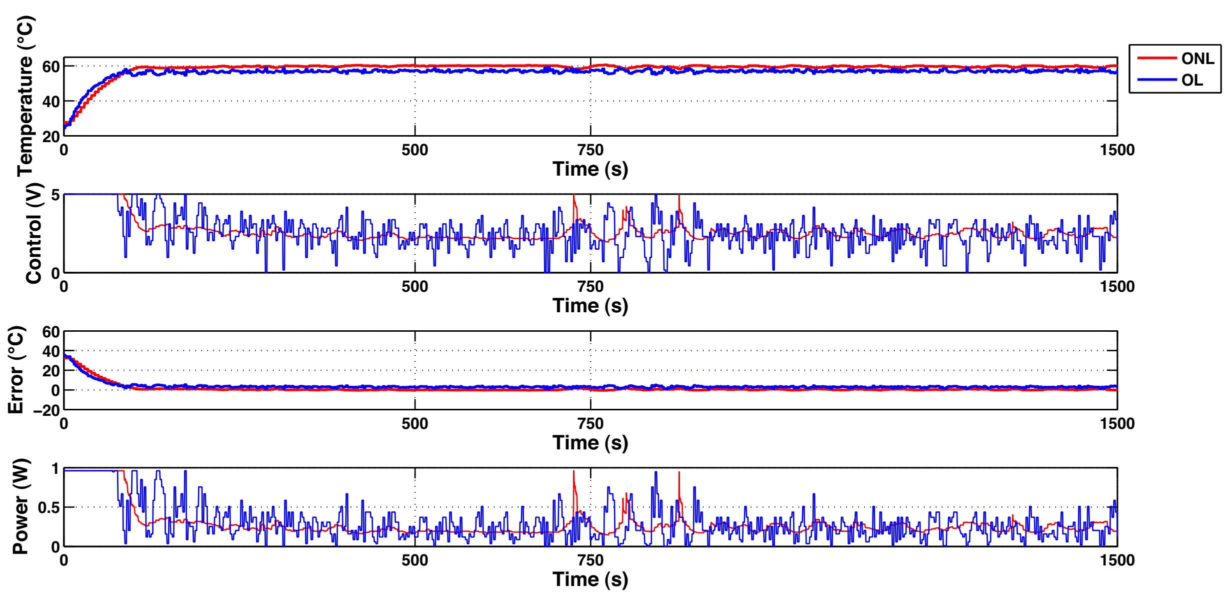

Fig. 10. Comparison of the system response with an optimal linear controller versus the optimal nonlinear controller.

Fig. 10 shows the experimental results, where we compare the performance of the optimal linear controller (with $Q = 1500$ and $R = 0.1$) versus the optimal nonlinear control synthesized in this work. We introduced a disturbance 750 s after the beginning of the experiment.

5. Conclusions

The optimal nonlinear control law which stabilizes asymptotically a class of nonlinear time delay systems is synthesized, provided a sufficient condition which guarantees that an explicit complete type functional $V(x_t)$ is a CLKF is satisfied. Unlike other approaches, our proposal is constructive and although some theoretical restrictions are imposed in the nonlinearities some experimental results demonstrate its efficiency. In spite of the fact that the condition guaranteeing that $V(x_t)$ is a CLKF introduce conservatism, it provides satisfactory performance results in the case of experimental tests, even when the system is perturbed. Moreover, the use of this type of control laws improves the quality of the products, which is currently investigated. Future work includes constrained controls.

Acknowledgements

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Appendix

In this section, we verify that the Hamilton-Jacobi-Bellman equation (23) is satisfied when the control law (20) is optimal and as we consider that the complete type functional $V(x_t^*)$ is the Bellman functional, we know from its derivative (12) the functionals $\Psi_0(x_t^*)$ and $\Psi_1^T(x_t^*)$, which are replaced into equations (22).

$$\begin{aligned}\left.\frac{dV(x_t^*)}{dt}\right|_{(8)}+q(x_t^*)+r(x_t^*)u^T(x_t^*)u(x_t^*) &= \Psi_0(x_t^*)+\Psi_1^T(x_t^*)\left(-\frac{1}{2}\frac{\Psi_1(x_t^*)}{r(x_t^*)}\right)+\Psi_1^T(x_t^*)\Psi_1(x_t^*)+\sqrt{\left[\Psi_0(x_t^*)\right]^2+\left[\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]^2}+r(x_t^*)\left(-\frac{1}{2}\frac{\Psi_1(x_t^*)}{r(x_t^*)}\right)^T\left(-\frac{1}{2}\frac{\Psi_1(x_t^*)}{r(x_t^*)}\right)\\ &= \Psi_0(x_t^*)-\frac{1}{2}\frac{\Psi_1^T(x_t^*)\Psi_1(x_t^*)}{r(x_t^*)}+\Psi_1^T(x_t^*)\Psi_1(x_t^*)+\sqrt{\left[\Psi_0(x_t^*)\right]^2+\left[\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]^2}+\frac{\Psi_1^T(x_t^*)\Psi_1(x_t^*)}{4r(x_t^*)}\\ &= \Psi_0(x_t^*)-\frac{1}{4}\frac{\Psi_1^T(x_t^*)\Psi_1(x_t^*)}{r(x_t^*)}+\Psi_1^T(x_t^*)\Psi_1(x_t^*)+\sqrt{\left[\Psi_0(x_t^*)\right]^2+\left[\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]^2}\end{aligned}$$

Substituting $r(x_t^*)$

$$\begin{aligned}&\Psi_0(x_t^*)-\frac{\Psi_1^T(x_t^*)\Psi_1(x_t^*)}{4\left(\dfrac{\left[\frac{1}{4}\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]}{\Psi_1^T(x_t^*)\Psi_1(x_t^*)+\Psi_0(x_t^*)+\sqrt{\left[\Psi_0(x_t^*)\right]^2+\left[\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]^2}}\right)}\\ &\quad+\Psi_1^T(x_t^*)\Psi_1(x_t^*)+\sqrt{\left[\Psi_0(x_t^*)\right]^2+\left[\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]^2}\\ &= \Psi_0(x_t^*)-\left(\Psi_1^T(x_t^*)\Psi_1(x_t^*)+\Psi_0(x_t^*)+\sqrt{\left[\Psi_0(x_t^*)\right]^2+\left[\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]^2}\right)\\ &\quad+\Psi_1^T(x_t^*)\Psi_1(x_t^*)+\sqrt{\left[\Psi_0(x_t^*)\right]^2+\left[\Psi_1^T(x_t^*)\Psi_1(x_t^*)\right]^2}\\ &= 0\end{aligned}$$

Then, the CLKF of system (8) and the optimal control law (20) satisfy the HJB equation (23) associated with the performance index (21).

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